A polynomial embbeding of pairs of partial orthogonal latin squares

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June 3, 2013

Abstract

We have shown that a pair of partial orthogonal latin squares of order t can be embedded in a pair of orthogonal latin squares of order $256t^4$ and all orders greater than or equal to $768t^4$. This the first polynomial embedding result of its kind.

1 Introduction and Definitions

Let $[n] = \{0, 1, \dots, n-1\}$ and N represent a set of n distinct elements. A non-empty subset P of $N \times N \times N$ is said to be a *partial latin square*, of order n, if for all $(x_1, x_2, x_3), (y_1, y_2, y_3) \in P$ and for all distinct $i, j, k \in \{1, 2, 3\}$,

$$x_i = y_i \text{ and } x_j = y_j \text{ implies } x_k = y_k.$$

We may think of P as an $n \times n$ array where symbol e occurs in cell (r, c), whenever $(r, c, e) \in P$. We say cell (r, c) is *empty* in P if, for all $e \in N$, $(r, c, e) \notin P$. The *volume* of P is $|P|$. If $|P| = n^2$, then we say that P is a *latin square*, of order n. We may use a latin square to define a binary operation; that is, for a latin square A, define $A(*)$ to be the *quasigroup* where for all $x, y \in N$,

$$x * y = z \text{ if and only if } (x, y, z) \in A.$$

Two partial latin squares P and Q, of the same order with the same non-empty cells, are said to be *orthogonal* if for all $r_1, c_1, r_2, c_2, x, y \in N$,

$$\{(r_1, c_1, x), (r_2, c_2, x)\} \subseteq P \text{ implies } \{(r_1, c_1, y), (r_2, c_2, y)\} \not\subseteq Q.$$

*This work was carried out during the scientist visit to Koç University supported by TUBITAK Visiting Scientist program 2221

†Author was also supported by Australian Research Council (grant number DP1092868)

‡This work was supported by Scientific and Technological Research Council of Turkey TUBITAK Grant Number:110T692

Example 1.1.

0	1	2	
2	0	1	3
3		0	
	2		1

0	2	1	
3	1	0	2
1		2	
	0		3

Figure 1: A pair of orthogonal partial latin squares of order 4

In 1960 Evans [2] proved that a partial Latin square of order n can always be embedded in some latin square of order t for every $t \geqslant 2n$. In the same paper Evan's raised the question as to whether a pair of finite partial latin squares, which are orthogonal, can be embedded in a pair of finite orthogonal latin squares.

It is known, (thanks to a series of papers by many authors, see for example [3]) that a pair of orthogonal latin squares of order n can be embedded in a pair of orthogonal latin squares of order t if $t \geqslant 3n$, the bound of $3n$ being best possible. Obtaining an analogous result for pairs of partial orthogonal latin squares has proved to be an extremely challenging problem. Lindner [7] showed that a pair of partial orthogonal latin squares can always be finitely embedded in a pair of orthogonal latin squares, however, there was no known method which obtains an embedding of polynomial order (with respect to the order of the partial arrays). In [4], Hilton et al. formulate some necessary conditions for a pair of orthogonal partial latin squares to be extended to a pair of orthogonal latin squares. Jenkins [5], considered the less difficult problem of embedding a single partial latin square in a latin square which has an orthogonal mate. His embedding was of order t^2.

More generally the study of orthogonal latin squares is a very active area of combinatorics (see [1]). It has been shown that a set of $n-1$ mutually orthogonal latin squares is equivalent to a projective plane of order n. (See [8] for detailed constructions). So the embedding of mutually orthogonal partial latin squares is equivalent to the embedding partial lines in finite geometries. Also the embedding of partial latin squares has a strong connection with the embedding block designs. For example many of the initial embeddings of partial Steiner triple systems used embeddings of partial idempotent latin squares (see for example [6]). It has also been suggested that the embeddings of block designs with block size 4 and the embeddings of Kirkman triple systems may make use of the embeddings of pairs of partial orthogonal latin squares (see [4]).

The present paper gives the first polynomial embedding result for a pair of partial orthogonal latin squares. Specifically we show that a pair of partial latin squares of order t can be embedded in a pair of latin squares of order at most $256t^4$. In many cases this order is much smaller and is of order $16t^2$.

We preface the discussion of our main result with some necessary definitions.

Let x_1, x_2, y_1, y_2 and z_1, z_2, be pairs of distinct element of N. The partial latin square $I = \{(x_1, y_1, z_1), (x_1, y_2, z_2), (x_2, y_1, z_2), (x_2, y_2, z_1)\}$ is termed an *intercalate*. If I is a subset of a latin square A, then $(A \setminus I) \cup I'$, where $I' = \{(x_1, y_1, z_1), (x_1, y_2, z_2), (x_2, y_1, z_2), (x_2, y_2, z_1)\}$ is a distinct latin square, of the same order. The partial latin square I' is said to be the *disjoint mate* of I. The form of the partial latin squares I and I' is displayed below. Here

the sideline lists the row labels and the headline lists the column labels.

I	y_1	y_2
x_1	z_1	z_2
x_2	z_2	z_1

I'	y_1	y_2
x_1	z_2	z_1
x_2	z_1	z_2

Two partial latin squares P and Q are said to be *isotopic*, if Q can be obtained by reordering the rows, or reordering the columns, or relabeling the symbols of P.

A set $T \subseteq A$, where A is a latin square of order n, is said to be a *transversal*, if

- $|T| = n$, and
- for all distinct $(r_1, c_1, x_1), (r_2, c_2, x_2) \in T$, $r_1 \neq r_2$, $c_1 \neq c_2$ and $x_1 \neq x_2$.

Note that a Latin square has an orthgonal mate iff it can be partitioned into disjoint transversals.

Let G denote the latin square corresponding to the abelian 2-group of order 2^m on the set $[2^m]$ and $G(\star)$ denote the corresponding quasigroup. The elements of the abelian 2-group will be labeled by $[2^m] = \{0, 1, 2, \dots, 2^m - 1\}$. Let 0 denote the identity element and recall that for all elements $x \in [2^m]$, $x \star x = 0$. Since $G(\star)$ is associative and commutative, to reduce unnecessary complexity, we will suppress the $\star$ notation and rewrite $x \star y$ as xy and in addition we will omit unnecessary brackets.

2 The basic construction

In this section we show that if P and Q are orthogonal partial latin squares, such that P contains no repeated symbols, then we may embed P and Q in orthogonal latin squares of order 2^m, for suitably chosen m.

We begin by noting that any partial latin square, of order t, can be embedded in a latin square of any order greater than or equal to $2t$ (see [2]).

Let P and Q be two orthogonal partial latin squares, of order t, volume v and based on the set of symbols $[t]$. Assume that P satisfies the property

- for all $x \in [t]$, $|\{(r, c, x) \in P \mid 0 \leqslant r, c \leqslant t - 1\}| \leqslant 1$.

Embed Q in a latin square B (quasigroup $B(\circ)$) of order 2^m, where m is a positive integer satisfying $2^{m-1} < 2t \leqslant 2^m$, and embed P in an $2^m \times 2^m$ array A, where the empty cells are filled with distinct symbols from the set $\{v, \dots, 2^{2m} - 1\}$. We will abuse notation and write $A(*)$ for the algebraic structure defined by the array A where, for all $r, c \in [2^m]$, $r * c = x$ if and only if $(r, c, x) \in A$ (note $x \in [2^{2m}]$). We make the following simple observation about A.

Lemma 2.1. *For all $r_1, r_2, c_1, c_2 \in [2^m]$, if $r_1 * c_1 = r_2 * c_2$, then $r_1 = r_2$ and $c_1 = c_2$.*

Proof. This follows directly from the fact that each symbol in $[2^{2m}]$ occurs in precisely one cell of A. □

We will construct arrays $\mathcal{A}$ and $\mathcal{B}$, of order 2^{2m}, which have the following properties:

- The row and columns of $\mathcal{A}$ and $\mathcal{B}$ are indexed by the ordered pairs $(x, y) \in [2^m] \times [2^m]$.
- The symbols of $\mathcal{A}$ will be chosen from the set $[2^{2m}]$.

- The symbols of $\mathcal{B}$ will be chosen from the set $[2^m] \times [2^m]$.

- The array $\mathcal{A}$ can be partitioned into $[2^m] \times [2^m]$ subarrays. That is, for $p, q \in [2^m]$, the intersection of rows $\{p\} \times [2^m]$ with columns $\{q\} \times [2^m]$ defines a subarray containing an isotopic copy of A where the rows and columns are permuted, but the symbols are left unchanged.

- The array $\mathcal{B}$ can be partitioned into $[2^m] \times [2^m]$ subarrays. That is, for each $p, q \in [2^m]$, the intersection of rows $\{p\} \times [2^m]$ with columns $\{q\} \times [2^m]$ defines a subarray containing an isotopic copy of B where the rows, columns and symbols have been permuted.

Thus $\mathcal{A}$ and $\mathcal{B}$ are defined as

$$\mathcal{A} = \{((p,r),(q,c), qr * pc) \mid (p,q,pq) \in G \wedge (r,c,r*c) \in A\}, \tag{1}$$

$$\mathcal{B} = \{((p,r),(q,c),(pq, p\,(qr \circ pc)) \mid (p,q,pq) \in G \wedge (r,c,r \circ c) \in B\}. \tag{2}$$

Recall that the $\star$ notation has been suppressed, so for instance, $p\,(qr \circ pc) = p \star ((q \star r) \circ (p \star c))$ in $G(\star)$.

The following lemma verifies that $\mathcal{A}$ and $\mathcal{B}$ are latin squares of order 2^{2m} and that $\mathcal{A}$ can be partitioned into transversals, one transversal for each symbol of $\mathcal{B}$.

Lemma 2.2. *Let P and Q be partial latin squares of order t and volume v. Let m be the smallest integer such that $2t \leqslant 2^m$. Embed P in an $[2^m] \times [2^m]$ array A, where each cell contains a distinct symbol from the set $[2^{2m}]$. Embed Q in a latin square B of order $[2^m]$. Let $\mathcal{A}$ be as defined in (1). Then $\mathcal{A}$ is a latin square of order 2^{2m} and can be partitioned into 2^{2m} transversal $\mathcal{T}_{(z,d)}$, where $(z,d) \in [2^m] \times [2^m]$. Let $\mathcal{B}$ be as defined in (2). Then $\mathcal{B}$ is a latin square of order 2^{2m}.*

Proof. It is not hard to see that $\mathcal{B}$ is the latin square obtained by taking the direct product of G with B and then permuting the rows and columns and relabeling the entries within the subsquares.

Assume $\mathcal{A}$ is not a latin square then, without loss of generality, there exists a row (column), say (p,r), which contains a repeated symbol. So assume that there exists columns (q,c) and (t,w), such that $qr * pc = tr * pw$. But $qr, pc, tr, pw \in [2^m]$, so Lemma 2.1 implies $qr = tr$ and $pc = pw$, and since $G(\star)$ is a group, $q = t$ and $c = w$. Thus $\mathcal{A}$ is a latin square of order $[2^{2m}]$.

Let $(z,d) \in [2^m] \times [2^m]$ and consider

$$\mathcal{T}_{(z,d)} = \{((p,r),(q,c), qr * pc) \in \mathcal{A} \mid pq = z, p\,(qr \circ pc) = d\}. \tag{3}$$

Let $((p,r),(q,c), qr * pc), ((s,u),(t,w), tu * sw)$ be distinct ordered triples of $\mathcal{T}_{(z,d)}$, so $pq = z = st$ and $p(qr \circ pc) = d = s(tu \circ sw)$. We wish to show that, respectively, the first coordinates, the second coordinates and the third coordinates are not equal.

Assume that this is not the case, so for instance assume $(p,r) = (s,u)$. Since $G(\star)$ corresponds to the abelian 2-group and $B(\circ)$ is a quasigroup, if $p = s$ and $pq = st$ then $q = t$, and if $r = u$ and $p(qr \circ pc) = s(tu \circ sw)$, then $qr \circ pc = qr \circ pw$ implying $c = w$. Thus $\mathcal{T}_{(z,d)}$ contains at most one cell from each row and similarly at most one cell form each column. In addition, for each $p \in [2^m]$ there exists a $q \in [2^m]$ such that $pq = z$, and given such a pair p, q, for each $r \in [2^m]$ there exists a $c \in [2^m]$ such that $p\,(qr \circ pc) = d$. So we may also deduce that $|\mathcal{T}_{(z,d)}| = 2^{2m}$.

We are left to consider the case where $(p,r) \neq (s,u)$, $(q,c) \neq (t,w)$, but $qr * pc = tu * sw$.

Since $qr, pc, tu, sw \in [2^m]$, Lemma 2.1 implies $qr = tu$ and $pc = sw$ and since $B(\circ)$ is a quasigroup, $qr \circ pc = tu \circ sw$.

But we have assumed that $p\,(qr \circ pc) = d = s(tu \circ sw)$ and so $p = s$. Since $pc = sw$, we obtain $c = w$. Also $pq = z = st = pt$ so $q = t$. But $qr = tu$, so $r = u$.

Thus the collection $\mathcal{T}_{(z,d)}$, $0 \leqslant d \leqslant 2^{2m}$, partitions $\mathcal{A}$ into disjoint transversals.

$\square$

Theorem 2.3. *Let P and Q be a pair of orthogonal partial latin squares of order t, such that each symbol of $[t]$ occurs in at most one cell of P. Then P and Q can be embedded in orthogonal latin squares $\mathcal{A}$ and $\mathcal{B}$ of order at most 2^{2m}, where m is a positive integer satisfying $2t \leqslant 2^m$.*

Proof. The above construction gives latin squares $\mathcal{A}$ and $\mathcal{B}$ satisfying (1) and (2). Then for all $(z, d) \in [2^m] \times [2^m]$, take $\mathcal{T}_{(z,d)}$ as defined in (3). Note that for fixed (z, d) the cells in $\mathcal{B}$ defined by $\mathcal{T}_{(z,d)}$ all contain the symbol (z, d) in $\mathcal{B}$. The result is now immediate. $\square$

3 Orthogonal trades in A

In the next section we would like to vary the above construction to allow for repeated symbols in the partial latin square P. We will do this by showing that $\mathcal{A}$ contains carefully selected intercalates which can be removed and replaced by disjoint mates to obtain a latin square with repeated elements in the subsquare defined by the intersection of rows $(0, r)$, $0 \leqslant r \leqslant 2^m$ and columns $(0, c)$, $0 \leqslant c \leqslant 2^m - 1$ (termed the top left corner), such that the new latin square is also orthogonal to $\mathcal{B}$.

In the previous section, in $\mathcal{A}$ the cell defined by row $(0, r_2)$ and column $(0, c_2)$ contained the symbol $r_2 * c_2$ and we want to be able to replace it by the symbol $r_1 * c_1$ which also occurs in the cell defined by row $(0, r_1)$ and column $(0, c_1)$. In this way we may introduce repeats into the top left corner in $\mathcal{A}$. Since $\mathcal{A}$ is a Latin square the repeated symbol $r_1 * c_1$ can not occur in the same row or column so we will require $r_1 \neq r_2$ and $c_1 \neq c_2$. Furthermore, as the new square should also be orthogonal to $\mathcal{B}$ we will require the corresponding entries in $\mathcal{B}$ are different, so $r_1 \circ c_1 \neq r_2 \circ c_2$.

In the next two lemmas we show that this switch is always possible provided $r_1c_1 \neq r_2c_2$ in the abelian 2-group.

We formalize these conditions as follows:

For positive integers $r_1, r_2, r_3, c_1, c_2, c_3, x \in [2^{m-1}]$, where $(r_1, c_1, x), (r_2, c_2, x) \in P$ and where appropriate $(r_3, c_3, x) \in P$, define Conditions C1 to C4 as follows.

C1. $r_1 < r_2 < r_3$.

C2. c_1, c_2, c_3 are all distinct.

C3. $r_1 \circ c_1, r_2 \circ c_2, r_3 \circ c_3$ are all distinct.

C4. $r_ir_j \neq c_ic_k$ whenever $i \neq j$, $i \neq k$ and $i, j, k \in \{1, 2, 3\}$.

Lemma 3.1. *Let $r_1, r_2, c_1, c_2 \in [2^{m-1}]$ and (r_1, c_1, x), (r_2, c_2, x) be a pair of triples in P which satisfy Conditions C1, C2, C3, and C4. Then $\mathcal{A}$ contains the following two intercalates, the*

union of which will be termed $I_A(r_1, c_1; r_2, c_2)$.

$I_A(r_1,c_1;r_2,c_2) \subset \mathcal{A}$	$(0, c_2)$	(r_1r_2, c_1)	$(r_1r_2(r_1 \circ c_1)(r_2 \circ c_2),$ $c_2(r_1 \circ c_1)(r_2 \circ c_2))$	$((r_1 \circ c_1)(r_2 \circ c_2),$ $c_1(r_1 \circ c_1)(r_2 \circ c_2))$
$R_1 = (0, r_2)$	$r_2 * c_2$	$r_1 * c_1$		
$R_2 = (c_1c_2, r_1)$	$r_1 * c_1$	$r_2 * c_2$		
$R_3 = (c_1c_2(r_1 \circ c_1)(r_2 \circ c_2),$ $r_2(r_1 \circ c_1)(r_2 \circ c_2))$			$r_1 * c_1$	$r_2 * c_2$
$R_4 = ((r_1 \circ c_1)(r_2 \circ c_2),$ $r_1(r_1 \circ c_1)(r_2 \circ c_2))$			$r_2 * c_2$	$r_1 * c_1$

Further in $\mathcal{A}$ these eight entries occur in eight distinct cells, each occurring in a different $2^m \times 2^m$ subsquare which contains a copy of A.

Proof. The existence of the eight entries is a straight forward computation following from the definition of $\mathcal{A}$.

Using the row labels R_1, R_2, R_3, R_4 as set out in the above table, the fact that $c_1 \neq c_2$ hence $c_1c_2 \neq 0$, implies $R_1 \neq R_2$, $R_1 \neq R_3$, $R_2 \neq R_4$ and $R_3 \neq R_4$. As $r_1 \neq r_2$, $R_1 \neq R_4$, $R_2 \neq R_3$. Similarly one can show that all the columns are distinct. Thus all four rows and columns are distinct from each other.

We are left to prove the entries occur in different subsquares. So assume that this is not the case and two of the entries of $I_A(r_1, c_1; r_2, c_2)$ occur in the same $2^m \times 2^m$ subsquare. Recall, that each such subsquare contains a copy of A and the symbols in the cells of A are all distinct. Also note that cells are in the same subsquare if and only if row wise they have the same first coordinates and column wise they have the same first coordinates. Now as $r_1r_2 \neq 0$, $c_1c_2 \neq 0$ and $r_1 \circ c_1 \neq r_2 \circ c_2$, the only possibility is that the fifth entry in the above table occurs in the top left subsquare, and so $c_1c_2(r_1 \circ c_1)(r_2 \circ c_2) = 0$ and $r_1r_2(r_1 \circ c_1)(r_2 \circ c_2) = 0$. These equations imply $r_1r_2c_1c_2 = 0$, which contradicts Condition C4 ($r_1r_2 \neq c_1c_2$). Thus all eight entries occur in different $2^m \times 2^m$ subsquares.

□

Lemma 3.2. *Let $r_1, r_2, c_1, c_2 \in [2^{m-1}]$ and (r_1, c_1, x), (r_2, c_2, x) be a pair of triples in P*

which satisfy Conditions C1, C2, C3, and C4. Then $\mathcal{B}$ contains the following eight entries.

$J_B(r_1, c_1; r_2, c_2) \subseteq \mathcal{B}$	$(0, c_2)$	$(r_1 r_2, c_1)$	$(r_1 r_2 (r_1 \circ c_1)(r_2 \circ c_2),$ $c_2 (r_1 \circ c_1)(r_2 \circ c_2))$	$((r_1 \circ c_1)(r_2 \circ c_2),$ $c_1 (r_1 \circ c_1)(r_2 \circ c_2))$
$(0, r_2)$	$(0, r_2 \circ c_2)$	$(r_1 r_2, r_1 \circ c_1)$		
$(c_1 c_2, r_1)$	$(c_1 c_2,$ $c_1 c_2 (r_1 \circ c_1))$	$(r_1 r_2 c_1 c_2,$ $c_1 c_2 (r_2 \circ c_2))$		
$(c_1 c_2 (r_1 \circ c_1)(r_2 \circ c_2),$ $r_2 (r_1 \circ c_1)(r_2 \circ c_2))$			$(r_1 r_2 c_1 c_2,$ $c_1 c_2 (r_2 \circ c_2))$	$(c_1 c_2,$ $c_1 c_2 (r_1 \circ c_1))$
$((r_1 \circ c_1)(r_2 \circ c_2),$ $r_1 (r_1 \circ c_1)(r_2 \circ c_2))$			$(r_1 r_2, r_1 \circ c_1)$	$(0, r_2 \circ c_2)$

Proof. Using the definition of $\mathcal{B}$ and the binary operation $\circ$ on B, we can calculate the entries directly.

□

Corollary 3.3. *Construct $\mathcal{A}$ and $\mathcal{B}$ as in* (1) *and* (2) *and let $r_1, r_2, c_1, c_2 \in [2^{m-1}]$ where (r_1, c_1, x), (r_2, c_2, x) be a pair of triples in P which satisfy Conditions C1, C2, C3, and C4. Then there exists a latin square $\mathcal{A}^*$ orthogonal to $\mathcal{B}$ that agrees with $\mathcal{A}$ everywhere except in the subsquare defined by the intersection of rows $(0, r)$ and columns $(0, c)$, $0 \leqslant r, c \leqslant 2^m - 1$, where the cell $((0, r_2), (0, c_2))$ is $r_1 * c_1$.*

Proof. By Lemmas 3.1 and 3.2 we know that there exist two orthogonal latin squares $\mathcal{A}$ and $\mathcal{B}$, for which $\mathcal{A}$ contains the configuration $I_A(r_1, c_1; r_2, c_2)$ which is the union of two intercalates. We may replace these intercalates by their disjoint mates to obtain $\mathcal{A}^*$. Further this trade preserves the orthogonality property and so the result follows. □

Note that (r_1, c_1) and (r_2, c_2) uniquely determine $I_A(r_1, c_1; r_2, c_2)$, and indeed the next lemma verifies that under certain condition we may construct collections of these configurations which are non-intersecting.

Lemma 3.4. *Assume $r_1, r_2, r_3, c_1, c_2, c_3 \in [2^{m-1}]$ and Conditions C1 through C4 are satisfied. Then there exists distinct $I_A(r_1, c_1; r_2, c_2)$ and $I_A(r_1, c_1; r_3, c_3)$ as described in Lemma 3.1 such that $I_A(r_1, c_1; r_2, c_2) \cap I_A(r_1, c_1; r_3, c_3) = \emptyset$.*

Proof. The existence of $I_A(r_1, c_1; r_2, c_2)$ and $I_A(r_1, c_1; r_3, c_3)$ follows from Lemma 3.1 and conditions C1 through C4, but we need to check that they are non-intersecting.

Considering $I_A(r_1, c_1; r_2, c_2)$ and $I_A(r_1, c_1; r_3, c_3)$. Since c_1, c_2, c_3 are all distinct, rows $(0, r_2)$, $(0, r_3)$, $(c_1 c_2, r_1)$, $(c_1 c_3, r_1)$ are all distinct. Further since $r_1 \circ c_1 \neq r_2 \circ c_2$ and $r_1 \circ c_1 \neq r_3 \circ c_3$, $(r_1 \circ c_1)(r_2 \circ c_2) \neq (r_1 \circ c_1)(r_3 \circ c_3)$ and so we may add rows $((r_1 \circ c_1)(r_2 \circ c_2), r_1(r_1 \circ c_1)(r_2 \circ c_2))$ and $((r_1 \circ c_1)(r_3 \circ c_3), r_1(r_1 \circ c_1)(r_3 \circ c_3))$ to this collection of distinct rows. Thus we have two remaining rows to check.

Thus we need to check that the following cases are not possible.

$$(c_1c_3(r_1 \circ c_1)(r_3 \circ c_3), r_3(r_1 \circ c_1)(r_3 \circ c_3)) = (0, r_2) \tag{4}$$
$$(c_1c_2(r_1 \circ c_1)(r_2 \circ c_2), r_2(r_1 \circ c_1)(r_2 \circ c_2)) = (0, r_3) \tag{5}$$
$$(c_1c_3(r_1 \circ c_1)(r_3 \circ c_3), r_3(r_1 \circ c_1)(r_3 \circ c_3)) = (c_1c_2, r_1) \tag{6}$$
$$(c_1c_2(r_1 \circ c_1)(r_2 \circ c_2), r_2(r_1 \circ c_1)(r_2 \circ c_2)) = (c_1c_3, r_1) \tag{7}$$
$$(c_1c_3(r_1 \circ c_1)(r_3 \circ c_3), r_3(r_1 \circ c_1)(r_3 \circ c_3)) = ((r_1 \circ c_1)(r_2 \circ c_2), r_1(r_1 \circ c_1)(r_2 \circ c_2)) \tag{8}$$
$$(c_1c_2(r_1 \circ c_1)(r_2 \circ c_2), r_2(r_1 \circ c_1)(r_2 \circ c_2)) = ((r_1 \circ c_1)(r_3 \circ c_3), r_1(r_1 \circ c_1)(r_3 \circ c_3)). \tag{9}$$

If (6) is true then

$$\begin{aligned} c_1c_3(r_1 \circ c_1)(r_3 \circ c_3) &= c_1c_2 \\ r_3(r_1 \circ c_1)(r_3 \circ c_3) &= r_1. \end{aligned}$$

This implies $r_1r_3 = c_2c_3$ which contradicts Condition C4, thus (6) is not possible, and similarly (7). If (8) is true then

$$\begin{aligned} c_1c_3(r_1 \circ c_1)(r_3 \circ c_3) &= (r_1 \circ c_1)(r_2 \circ c_2) \\ r_3(r_1 \circ c_1)(r_3 \circ c_3) &= r_1(r_1 \circ c_1)(r_2 \circ c_2). \end{aligned}$$

But this implies $r_1r_3 = c_1c_3$ which contradicts Condition C4, thus (8) is not possible, Similarly (9) is not possible.

If (4) is true then

$$\begin{aligned} c_1c_3(r_1 \circ c_1)(r_3 \circ c_3) &= 0 \\ r_3(r_1 \circ c_1)(r_3 \circ c_3) &= r_2. \end{aligned}$$

But this implies $c_1c_3 = r_2r_3$ once again contradicting Condition C4, thus (4) is not possible and similarly (5) is not possible.

Thus the result follows.

□

4 General embedding

Let P and Q be two orthogonal partial latin squares, of order t, volume v and based on the set of symbols $[t]$. Note that there are no restrictions on P. We begin by replacing entries in some cells of P to obtain a partial latin square that contains no repeated symbols. We then embed this new partial latin square and finally interchange intercalates to obtain orthogonal latin squares which contain P and Q.

For each $x \in [t]$, define

$$u_x = \min\{u \mid (u, v, x) \in P\}.$$

So that u_x is the first row in P that contains x. Let m be the smallest integer such that $2^m \geqslant 2t$.

Theorem 4.1. *Let P and Q be a pair of orthogonal partial latin squares of order t, on the set of symbols $[t]$ such that for all $r_1, r_2, c_2, c_2 \in [2^{m-1}]$ and for all incidence of $(r_1, c_1, x), (r_2, c_2, x) \in P$ Conditions C1 to C4 are satisfied. Then P and Q can be embedded in orthogonal latin square $\mathcal{A}^*$ and $\mathcal{B}$ of order at most 2^{2m}, where m is the smallest positive integer such that $2t \leqslant 2^m$.*

Proof. Construct a new partial latin square P^* as follows. If there exist $(r, c, y) \in P$, place (r, c, x) in P^*, where

$$x = \begin{cases} y, & \text{if } r = u_y \\ w, & \text{otherwise, where } w \in \{t, \ldots, v-1\} \text{ and } w \text{ has not previously been used.} \end{cases}$$

Thus P^* and P will have the same filled cells, however the cells of P^* will contain distinct entries.

Respectively embed P^* and Q, in the arrays A and B of order 2^m as described in Section 2. Let $A(*)$ and $B(\circ)$ be the respective algebraic structures. Use Theorem 2.3 to embed A and B in orthogonal latin square $\mathcal{A}$ and $\mathcal{B}$, of order at most 2^{2m}.

If there exists $(u_y, v_y, y), (r_2, c_2, y) \in P$ such that $r_2 \neq u_y$, then let $r_1 = u_y$, $c_1 = v_y$ and consequently $r_1 * c_1 = y$. Thus let $((0, r_1), (0, c_1), r_1 * c_1)$ be the entry in $\mathcal{A}$ corresponding to $(u_y, v_y, y) \in A$ and $((0, r_2), (0, c_2), r_2 * c_2)$ be the entry in $\mathcal{A}$ corresponding to $(r_2, c_2, w) \in A$, where $w \neq y$.

Since $r_1 r_2 \neq c_1 c_2$, Corollary 3.3 implies there exists orthogonal latin square $\mathcal{A}^*$ orthogonal to $\mathcal{B}$, such that $((0, r_2), (0, c_2), r_1 * c_1) \in \mathcal{A}^*$.

Finally assume that there exists $(u_y, v_y, y), (u_z, v_z, z), (r_2, c_2, y), (r_3, c_3, z) \in P$ such that $r_2 \neq u_y$ and $r_3 \neq u_z$. If $y \neq z$ then $I_A(u_y, v_y; r_2, c_2)$ and $I_A(u_z, v_z; r_3, c_3)$ can be constructed as in Lemma 3.1, and respectively, they contain the distinct symbols $u_y * v_y, r_2 * c_2$ and $u_z * v_z, r_3 * c_3$, and so do not intersect. Thus the corresponding intercalates can be removed and replaced by their disjoint mates to obtain the required embedding.

If $y = z$, then $u_y = u_z$ and $v_y = v_z$ and there exists $(u_y, v_y, y), (r_2, c_2, y), (r_3, c_3, z) \in P$ such that $r_2 \neq u_y$ and $r_3 \neq u_y$. Now since Conditions C1 to C4 are satisfied Lemma 3.4 implies $I_A(u_y, v_y; r_2, c_2) \cap I_A(u_y, v_y; r_3, c_3) = \emptyset$. So for every instance of $(r, c, y) \in P$, such that $r \neq u_y$, we may trade intercalates in $\mathcal{A}$ to obtain orthogonal latin squares $\mathcal{A}^*$ and $\mathcal{B}$ which contain respectively an embedded copy of the orthogonal partial latin squares P and Q. $\square$

The above embedding works provided Conditions C1 to C4 are satisfied. If this is not the case then we begin by expanding the columns of P and Q and relabeling the symbols to ensure that these conditions are satisfied and then we may apply Theorem 4.1.

So let P and Q be two orthogonal partial latin squares, of order t, volume v and based on the set of symbols $[t]$. Let m is a positive integer such that $t \leqslant 2^m$. If there exists $r_1, r_2, r_3 \in [2^{m-1}]$, $(r_1, c_1, x), (r_2, c_2, x), (r_3, c_3, x) \in P$ such that Conditions C1 to C4 are not satisfied then construct new partial latin squares P_0 and Q_0 of order 2^{2m} as follows:

$$\begin{aligned} P_0 =& \{(r, c + c \cdot 2^m, e) \mid (r, c, e) \in P\} \\ Q_0 =& \{(r, c + c \cdot 2^m, e) \mid (r, c, e) \in Q\}. \end{aligned}$$

Here $+$ and $\cdot$ are the normal operations of addition and multiplication on the positive integers. Note that P_0 and Q_0 are of order $t2^m$. Further, since $(r_1, c_1, x), (r_2, c_2, x), (r_3, c_3, x) \in P$ and P is a partial latin square Conditions C1 and C2 must be satisfied. Since Q is an orthogonal mate to P Condition C3 must be satisfied. Let $i, j, k \in \{1, 2, 3\}$, $i \neq j$ and $i \neq k$. $(r_1, c_1 + c_1 \cdot 2^m, x), (r_2, c_2 + c_2 \cdot 2^m, x)$ and $(r_3, c_3 + c_3 \cdot 2^m, x) \in P_0$, then in the abelian 2-group $G(\star)$ of order 2^{2m}, since $r_1, r_2, r_3 \in [2^{m-1}]$, $r_i \star r_j$ belongs to a subgroup of order 2^m, but $c_i + c_i \cdot 2^m$ and $c_k + c_k \cdot 2^m$ are not elements of this subgroup. Indeed they belong to distinct cosets of the quotient group. Thus

$$(c_i + c_i \cdot 2^m)(c_k + c_k \cdot 2^m) \neq r_i r_j.$$

and so Condition C4 is satisfied. Consequently the partial latin squares P_0 and Q_0 satisfy the conditions of Theorem 4.1 and can be embedded in orthogonal latin squares of order 2^{4m}, validating the proof of our main theorem that is stated below. Note that there exists a copy of P and Q on the intersection of the rows $\{0\} \times \{0, ...2^m - 1\}$ and columns $\{0\} \times \{c + c2^m | c \in [2^m - 1]\}$ in $\mathcal{A}^*$ and $\mathcal{B}$, respectively.

Theorem 4.2. *Let P and Q be a pair of orthogonal partial latin squares of order t, on the set of symbol of $[t]$. Then P and Q can be embedded in orthogonal latin square $\mathcal{A}^*$ and $\mathcal{B}$ of order at most 2^{4m}, where m is the smallest positive integer such that $2t \leqslant 2^m$.*

Corollary 4.3. *Let P and Q be a pair of orthogonal partial latin squares of order t, on the set of symbol of $[t]$. Further P and Q can be embedded in orthogonal latin square $\mathcal{A}^*$ and $\mathcal{B}$ of order at most $256t^4$ and all orders greater than or equal to $768t^4$.*

Proof. By the above theorem the embedding has order at most 2^{4m} where m is the smallest positive integer such that $2t \leqslant 2^m$. Then $2^{m-1} < 2t \leqslant 2^m$ so $2^m < 4t$. Hence $2^{4m} < 256t^4$. Since a pair of orthogonal latin squares of order n can be embedded in a pair of orthogonal latin squares of all orders greater than or equal to $3n$ [3] the result follows. $\square$

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